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Recent Developments in the Foundation of Mathematics and Fundamental Analytic Flaws in Gödel's Incompleteness Thesis

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Abstract

Philosophy of Mathematics deals with foundation of mathematics. Russell's Logicist Program and Hilbert's Formalist Program are two attempts Philosophy of Mathematics deals with foundation of mathematics, though practicing mathematicians seldom bother about the nature of mathematical objects or about ground for mathematical truths. However, it's important to know what makes mathematical truths necessary? To understand that we need to go through the foundational issues pertaining mathematics. Russell's Logicist Program and Hilbert's Formalist Program are two attempts to secure the foundation of mathematics. For about 80 years including these projects, foundational programs for mathematics seemed to have lost its significance in the face of Gödel's Incompleteness Thesis, considered as a paradigm in mathematics. However, a recent development in the principles of mathematics witnessed some fundamental flaw in Gödel's Thesis, and not only has given oxygen to foundational research in mathematics, also seems to have revolutionized the nature of analysis in mathematics.

Keywords:

: Foundation of Mathematics, Peano Arithmetic, Number Theory, Ligicism, Formalism, Analysis of Class and Number, Mathematical Logic, Cantor's Paradox, Well Formed Formula

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1. Introduction

Russell's Logicist Program and Hilbert's Formalist Program are two attempts to secure the foundation of mathematics. For about 80 years including these projects, foundational programs for mathematics seemed to have lost its significance in the face of Gödel's Incompleteness Thesis, considered as a paradigm in mathematics. However, a recent development in the principles of mathematics witnessed some fundamental flaws in Gödel's Thesis, and not only has given oxygen to foundational research in mathematics, also seems to have revolutionized the nature of analysis in mathematics. In the **Introduction** of the present paper, we shall discuss the nature of philosophy of mathematics and the significance of securing foundation of mathematics. In the **Second section,** we shall discuss briefly the salient feature of Russell'sLogicist Program. In the **Third section**, we shall discuss Russell's theory of number, defined in terms of the notion of *class*as exposed in *Principia Mathematica*; and we shall show how does it abides by Peano Arithmetic. In the **Forth section**, we shall briefly discuss Gödel's incompleteness Thesis and the fundamental flaws in his account. In **Conclusion**, we shall discuss the present scenario in the foundational approaches in mathematics.

Let us start with few quarries':
Q1.) What is the significance of raising foundational questions in present day?
Q2.) What is the significance of raising the issue of foundation of mathematics?
Q3.) What is the significance of considering a logicist account in providing a foundation of mathematics?

Q4.) What is the significance of Russell's logicist program in providing a foundation of mathematics?

Foundationalism, traditionally, is a view about the structure of justification of a knowledge claim, or of a system of belief. The foundationalist thesis, in short, is that all knowledge or a system of true beliefs rests ultimately on a foundation of non-inferential knowledge. Traditionally, it is necessary for any knowledge claim to justify the ground and nature of its truth. The question of justification led philosophers, for long, to look for a foundation — the rock-bottom of a set of true beliefs claimed as knowledge.

But, in the middle of the 19th century, due to naturalistic turn (led by Quine[1] [W.V. Quine (1966)], Putnam[2] [H. Putnam (1967)], and others) and anti-foundationalist approach (led by Sellars[3] [Sellars (1963)], BonJour[4] [BonJour(1985)], Bergmann[5] [Bergman (2006)], Sosa[6] [Sosa (2003)] and others) in epistemology, question of explanation got priority over/replaced the question of justification and thus, the issue of foundation. But, in spite of that, there remain some fundamental questions that no scientific enquiry can escape. Some of these are as follow

Q1.} What is the theory under consideration about? Or, What is the subject matter of the theory?

Q2.} What is the ground of the necessity (Physical or Logical) of the truth of the theory?

Q3.} How is the theory applicable to external reality?

These unavoidable questions are such that we have to go through the foundation of that theory (i.e., go through the issue of justification), in order to address these questions. In fact, this approach gives rise to a movement, according to Shapiro (1991)[7], called securing *foundation without foundationalism*.

Mathematics deals with numbers, though practicing mathematicians seldom bother about nature of numbers as mathematical object and/or ground of mathematical truths. These are supposedly concerns of philosophers of mathematics. However, such issues are, as we have seen, very important for any scientific enquiry. If mathematics is regarded as a science, then philosophy of mathematics can be regarded as a branch of the philosophy of science, next to disciplines such as philosophy of physics and philosophy of biology. However, because of its subject-matter, philosophy of mathematics occupies a special place in the philosophy of science. Whereas, natural sciences investigate entities that are located in space and time, it is not at all obvious that this is also the case with respect to the objects that are studied in mathematics. In addition to that, the methods of investigation of mathematics differ markedly from the methods of investigation of natural sciences. Whereas natural sciences acquire general knowledge using inductive methods, mathematical knowledge is acquired by deduction from basic principles. The status of mathematical knowledge also differs from the status of knowledge in natural sciences. The theories of natural science appear to be less certain and are more open to revision in contrast to mathematical theories. For these reasons, mathematics poses problems of a quite distinctive kind for philosophy. Therefore, philosophers have accorded special attention to ontological and epistemological questions concerning mathematics, along with foundational questions.

Well-known foundational movements in mathematics started due to the following crisis [Mukhopadhyay, A. K. (2011a)][8]:

1. Emergence of a number of non-Euclidean geometrical systems.
2. Emergence of a host of paradoxes.

Three fundamental notions in mathematics — *infinity*, *the infinitesimal* and *continuity* – seemed inherently paradoxical [Mukhopadhyay, A. K. (2011a)].

By the middle of the 19th century, these problems at the heart of pure mathematics have inspired, some German mathematicians, to provide mathematics with a rigorous foundation. As a result of the work of Weierstrass (1815 - 1897), Dedekind (1831 - 1916), and Cantor(1845 - 1918), the notion of an infinitesimal has been banished, real numbers has been provided with a logically consistent definition, continuity has been redefined and, more controversially, a whole new branch of arithmetic, namely, *transfinite arithmetic*, has been invented which addresses itself to the paradoxes of infinity.

But, the picture of mathematical realm that results from the above mentioned works becomes even more bewildering than that emerges from the mathematics of Newton and Leibniz.

Cantor, for example, tries to solve these problems by his (naive) *set theory* which itself invites troubles. Within his system Cantor proves that *there is no such thing as the highest cardinal number*. In Cantor's system, cardinal numbers belong to sets, and there are finite as well as infinite cardinals. The set of natural numbers, for example, has an infinite cardinal to which Cantor assigned the symbol № —and so does the set of real numbers. But, Cantor has a proof that the set of real numbers has more members than the set of natural numbers. This proof works by demonstrating first that the natural numbers are a proper subset of the reals, and second, that the reals cannot be put into one-to-one correspondence with the naturals. It follows, Cantor argues, that the set of reals is bigger than the set of naturals. Cantor also has a proof that, in general, a set has fewer members than its *power set* (i.e., the set of its subset). If a set has *n* members, then there will be 2^n subset of it, and $2^n$is always greater than *n*.Putting these two proofs together, Cantor concludes that the set of real numbers has the cardinality 2№. From here, Cantor constructs an entire hierarchy of different finite and infinite numbers that can be continued indefinitely. Cantor conceives of *transfinite numbers* as representing the cardinality of an infinite set, just as a natural (finite) number is the cardinality of a finite set. The cardinality of the set of all natural numbers is the first transfinite cardinal number. In this process, Cantor conceives of the set of all sets. According to Cantor, there cannot be a *greatest* infinite cardinal number because, whatever cardinal number one may take, one can always have a larger one by forming its power set. But, for the case of *universal set*, i.e., the set of all sets, its cardinality should be the greatest transfinite cardinal number that could exist. However, since, for Cantor, the set of all subsets of a given set must have a greater cardinality than the set itself, then the power set (the set of all subsets) of Universal set (the set of all sets), which is an infinite set, must have greater cardinality than the universal set; that is, there must be a greater transfinite cardinal number than the greatest! Surely, this is a contradiction.

Now, these are the main reasons that initiated the 19th Century foundational movements in mathematics. Again, as the different foundational moves have taken place, new kinds of puzzles and paradoxes emerge and foundational movement progresses up until there is a naturalistic turn in the philosophy. But, after a break again there were foundational enquiries that we have already mentioned.

Although the significance of raising the foundational issues in mathematics is, however, different from the reasons for raising them traditionally, these issues also, should not be avoided.

If we translate the fundamental question that we have mentioned earlier into mathematical context, we will have the following —

Q1) What is mathematics about? Or, what is the nature of number as mathematical object?

Q2) What is the source of mathematical necessity? Or, what is the source of the necessity of mathematical truths?

Q3) How is mathematics applicable to the external world?

Since, mathematics too is a scientific theory, any mathematician, whether a practitioner or a theoretician, is bound to address these questions.

One of the significant projects in securing the foundation of mathematics is Russell's Logicist Program, initially proposed in his *The Principles of Mathematics*, and later developed in *Principia Mathematica*.

2. Salient Feature of Russell's Logicist Program:

Russell observes that the main reason regarding the above mentioned inconveniences in mathematics are [Mukhopadhyay, A. K. (2011a)]-

1. Commitment to the existence of class / set.
2. Unrestricted allowance for class/set formation.

To get rid of these Russell introduced two devices [Mukhopadhyay, A. K. (2011a)]-

1. Theory of Incomplete Symbols (to remain non-committal to the existence of class) [9].
2. Theory of Logical Types (to impose a rule for class formation0 [Mukhopadhyay, A. K. (2015)].

Theory of Incomplete Symbols (a more general version of his theory of descriptions) introduces a new and powerful method of analysis that is in many ways guiding force to advanced analytic philosophy. Existence is treated here as a property of propositional function. It follows that the kinds of objects which are said to exist will depend on the kinds of propositional functions which are said to be satisfied; and this is the source of Quine's celebrated dictum that *'to be is to be the value of a variable'*. An ascription of existence can not significantly be coupled with the use of a logically proper name.

Theory of Logical Types gave the grammar of the logical language in which Russell wished explain mathematical notions. The primary objects or individuals (i.e., the given things not being subjected to logical analysis) are assigned to one type, say, *type 0.* Propositional functions applicable to individuals are assigned to.

type 0.	**arnab**	**('a')**
type 1.	**'arnab is a fool'**	**('*f*(a)')**
type 2.	**"'arnab is a fool' is true"**	**('*g*(*f*a)')**

To exclude impredicative definitions within a type, the types above type 0 are further divided into orders.

Basic theses of Russell's Program as expounded in *Principia Mathematica* can be summarized as follows[10]:

- ✓ Logic and Language-
  - Theory of descriptions constitutes the core of the core of the general theory of meaning.
  - Distinction between logical form and grammatical form of linguistic expression.
  - Language for logic has predicate variables with order \ type indices and individual variables.
  - Logic is the general theory of the structure.
  - Material implication and other logical connectives are not relation signs.
  - No denoting concepts.
  - No bridge between categorical logic and the new quantification theory (variables of the new quantification theory taken as primitive).
  - No substitutional theory of propositions emulating simple types of attributes / classes.
  - Axiom of reducibility and axiom of infinity are admitted.
  - Recursive definition of 'truth' and 'falsehood' justifying order component of the order \ type indices on predicate variables.

- ✓ Ontology-
  - Abolition of propositions (elementary and general) as single independent entities, instead the multiple relation theory of judgment is admitted
  - Admission of universals (type free with both a predicable and individual nature).
  - No non-existent objects since truth conditions for sentences with names can be given by descriptions.
  - Abolition of classes as entities.
  - Abolition of numbers as entities.
- ✓ Epistemology-
- Principle of acquaintance upheld, and acquaintance with universals, sense-data and the subject admitted.

The axiomatic basis of the PM system –

The set of primitive symbols of the language of *PM system (in the 1st Ed.) includes the following:*

i) *the symbols '~' and* 'V' for negation and disjunction respectively;

ii) $x^0_1, x^0_2, \ldots, x^0_n$ are called individual variables (informally, x, y, z, each with appropriate type index which in the present case is the symbol for the lowest type, i.e. type 0);

iii) $x^t_1, x^t_2, \ldots, x^t_n$,where t is not 0, are called functional variables (informally, Ø, ψ,χ, each with appropriate type index which in this case is greater than the type 0);

iv) 'for all' ('∀') and 'there exists at least one' ('∃') for universal quantifier and existential quantifier respectively;

v) The symbols for punctuation – '(', ')', '[', ']', '{', '}' (in *Principia*, also '**.**').

A type symbol of simple type theory is defined recursively as follows:

i) 0 is a type symbol.

ii) If $t_1, t_2, \ldots, t_n$ are type symbols, then $(t_1, t_2, \ldots, t_n)$ is a type symbol.

iii) There are no other type symbols.

In accordance with *Principia*'s ramified type theory, the notion of the order of a simple type symbol is further defined as:

i) The type symbol 0 has order 0.

ii) A type symbol $(t_1, t_2, \ldots, t_n)$ has order n+1 if the highest order of the type symbols $t_1, t_2, \ldots, t_n$ is n.

The atomic formulae are of the form:

$$Ø^{(t1,\, t2,\, \ldots,\, tn)}\,(x^{t1}_1, x^{t2}_2, \ldots, x^{tn}_n)$$

The set of formulae constitutes the smallest set F containing all atomic well formed formulae (wffs) such that if A, B, C are quantifier free formulae in F and x^t is an individual variable free in C, then ~ (A), (A V B), (∀x) C, and (∃x) C are in F too.

Some of the definitions of new symbols in the language of the *PM system (in the 1st Ed.) are the following:*

i) $(x^t = y^t) = df\ (\forall Ø^{(t)})\,[Ø^{(t)} x^t \equiv Ø^{(t)} y^t]$

ii) $(A \supset B) = df\ (\sim A \vee B)$.

iii) $(A \,\&\, B) = df\ \sim(\sim A \vee B)$.

iv) $(A \equiv B) = df\ [(A \supset B) \,\&\, (B \supset A)]$

Taking p, q, r as schematic for quantifier free formulae, and A, B, C as schematic for any formula, the axiom schema are as follows:

*1.2 (p V p) ⊃ p
*1.3 q. ⊃ (p V q)
*1.4 (p V q) ⊃ (q V p)
*1.5 [p V (q V r)] ⊃ [q V (p V r)]
*1.6 (q ⊃ r) ⊃ [(p V q) ⊃ (p V r)]
*9.1 A $[y^t \mid x^t]$ ⊃ (∃ x^t) A
where y^t is free for x^t in A.
*9.12 A$[y^t \mid x^t]$ V A $[z^t \mid x^t]$ ⊃ (∃ x^t)A,
where $y^t$and z^t are free for x^t in A.

The inference rules of the *PM system (in the 1st Ed.) are:*

i) from A and A ⊃ B, infer B (Modus Ponens).
ii) from A, infer (∀ x^t)A (Universal Generalization).

Although, negation and disjunction retain their essential features as truth functions, when extended to wffs with bound variables, Whitehead and Russell have defined negation and disjunction for generalized formulae through the following definitions:

*9.01 ~(∀ x^t) Ax^t = df (∃ x^t) ~Ax^t
*9.02 ~ (∃ x^t) Ax^t = df (∀ x^t)~Ax^t
*9.03 [(∀ x^t) Ax^t V p] = df (∀ x^t)(Ax^t V p)
*9.04 [p V (∀ x^t) Ax^t] = df (∀ x^t)(p V Ax^t)
*9.05 [(∃ x^t)Ax^t V p] = df (∃ x)(Ax V p)
*9.06 [p V (∃ x^t) Ax^t] = df (∃ x^t) (p V Ax^t)
*9.07 [(∀ x^t) Ax^t V (∃ y^v) By^v] = df (∀ x^t)(∃ y^v) (Ax^t V By^v)
*9.08 [(∃ y^v) Ay^v V (∀ x^t) Bx^t] = df (∀ x^t)(∃ y^v)(Ay^v V Bx^t).

The above is a description of the axiomatic basis of the logic of PM *system (as in the 1st Ed.). An alternative equivalent axiomatization specifically, regarding the quantificational part of PM has also been given in *10 of the 1st edition of Principia Mathematica*, vol.1, where universal quantifier is taken as primitive and existential quantifier is defined in terms of it. The set of axioms incorporates the standard axiom for universal specification, and rules of inference are as in * 9 in *Principia*.

In the PM system Whitehead and Russell define symbols for classes contextually in the following way –

* 20.01] $f\{\hat{y}^t(\psi^{(t)} y^t)\}$ = df. (∃ $Ø^{(t)}$) (∀ x^t) [($Ø^{(t)}x^t$≡$\psi^{(t)}x^t$)&f ($Ø^{(t)}$)],

where *f* is a given function.

The following is a contextual definition of symbol for class of classes –

*20.08 f {$\hat{a}^{(t)}$ ($\psi^{((t))}$ $a^{(t)}$)} = df. (∃ $Ø^{((t))}$)($a^{(t)}$)[($\psi^{((t))}$ $a^{(t)}$)≡ $Ø^{((t))}a^{(t)}$) &f ($Ø^{((t))}$)],

where *f* is a given function and $a^{(t)}$ is schematic for some $\hat{y}^{(t)}$ ($\chi^{(t)}(y^t)$).

3. Russell's Theory of Classes and Number:

In the *Principia Mathematica*, Russell, along with Whitehead, maintains that the theory of classes, although provides a notation to represent them (classes), it avoids the assumption that there are such things as classes. [Russell, B. and Whitehead, A.N.(1910)]. Russell seeks to give a definition of symbols for classes on a similar line as definitions of descriptions, taking them as incomplete symbols. Such definition will assign meaning (i.e., truth or falsity) to statements in which words or symbols representing classes occur. Such a definition will assign meaning to statements containing class-symbols, eliminating all mention of classes from a proper analysis of those statements. If this becomes possible then Russell would say that symbols for classes are mere conveniences, like descriptions, they are 'logical fictions' [Russell, B (1919)].

Class as Incomplete Symbols[11]:

In the *Principia Mathematica* this technique of deriving an extensional function from a function of a given function is presented in the form of the following definition:

f ({z: ψz}) = (∃Ø)[(x) (Øx≡ψx). f {z: Øz}]   Df   [*20.01]

The definition *20.01 in the *Principia Mathematica* actually stipulates the condition when a statement asserting some 'propositional function' ψx can be made.

The condition is that there must be a predicative function Ø x formally equivalent to ψx such that an assertion f of Øx is meaningful (i.e., true/false). The equality between the two formally equivalent propositional functions is their identical extension, which renders the assertion f of ψx to be considered as the assertion f of Ø x. Thus, f of ψx can be considered as an assertion of this common extension. For the sake of convenience this extension is called 'the class determined by the propositional function (condition) ψx'. In this way an assertion *f* of ψx becomes an assertion *f* of the class determined by the propositional function (condition) ψx.

In the above, by a predicative function what is meant is — *a function of one variable which is of the next order above that of its argument, i.e., the lowest order compatible with the order of that argument*.

The definition *20.01 is in fact the definition of a class in use. This definition basically effectuates reduction (translation) of statements nominally about classes to statements about their defining conditions [Russell, B. (1919)].

Requirements of 'Class' [Mukhopadhyay, A. K. (2015)]**:**

Now, if a symbol is to serve as a class it must fulfil the following conditions [Russell, B. (1919)] :

i) A class is always determined by a predicative propositional function, and that a predicative propositional function must determine an appropriate class.

ii) Two formally equivalent propositional functions determine the same class and two propositional functions that are not formally equivalent to each other must determine two different classes. This is known as *the principle of extensionality for classes*.

iii) There must be a mechanism for defining not only classes, but classes of classes also.

Russell has shown in *Principia Mathematica* that classes of classes too have all formal properties of classes of individuals. We will see shortly that numbers have been defined by Russell as classes of classes which are similar to each other.

iv) The question whether a class is a member of itself or not, will not be entertained in the theory of classes. Type theory takes care of this.

v) Mathematical induction involves reference to all natural numbers less than/equal to a certain, number k. This brings in the notion of universal class, i.e., class of all individuals, class of all classes etc.

However, unless all the elements of a so-called universal class are of the same logical type, questions regarding the legitimacy of a universal class will continue to be raised. In the theory of classes, as proposed in *Principia Mathematica*, the class consisting of all elements of a given type is called a universal class, the class determined by the 'propositional function' (condition) 'x = x', and is symbolically represented by 'V'.

Thus, $V = \{x : x = x\}$ [24.01]

The null class, symbolically represented by 'Λ', is the complement of V, or,

$\Lambda = -V$ [*24.02].

In this context, in *Introduction to Mathematical Philosophy* Russell defines unit class by saying — A class α is said to be a "unit" class if the propositional function "'x is an α' is always equivalent to 'x is c' " (regarded as a function of c) is not always false, i.e., in more informal language, if there is a term c such that x will be a member of α when x is c but not otherwise. [Russell, B. (1919)]. Taking 'x is an α' as 'Øx', symbolically we may put it as follows —

$$\alpha = (\exists c)\,(x)\,(Øx \equiv x = c) \text{ Df.}$$

In general, a class α is the collection of all those entities x's that satisfy a predicative propositional function ϕz.

The predicative functions are brought in to ensure that the hierarchy of logical types is strictly maintained in formation of a class and also in formation of any statement about classes.

The definition 20.01 is a definition in use of an expression 'α' such that $\alpha = \{z : \psi z\}$; in other words, 20.01 is a definition of ''the class determined by the propositional function (condition) ψz'', whenever there is a predicative function Øz equivalent to ψz, and '$f\{z : Øz\}$' is significant.

However, in *Principia*, there is a separate definition of 'class of classes', not only because the notion of number is defined in terms of the notion of 'class of classes', but also because of some deeper reason. We will discuss about it in the next section.

The following is the definition in use of 'class of classes'.

$f\{\alpha : \psi\alpha\} = (\exists\phi)[\,(\alpha)\,(\psi\alpha \equiv \phi\alpha).f(\phi)]$ [*20.08]

The above definition actually stipulates the condition when a statement involving the propositional function 'ψα', where α is a class, can be made. The condition is that there must be a predicative function 'ϕβ', formally equivalent to 'ψβ' such that an assertion of '$f(\phi\alpha)$' is meaningful (i.e., true ro false).

In *Principia Mathematica*, classes of individuals are proved to satisfy certain properties like,

$(x)\,(\psi x \equiv Øx) \equiv [\{z : \psi z\} = \{z : Øz\}]$ (*20.15),

$[\{z : \psi z\} = \{z : Øz\}] \equiv (x)\,[x \in \{z : \psi z\} \equiv x \in \{z : Øz\}]$ (*20.31),

$[\{z : Øz\} = \{ z : \psi z\}] \rightarrow [f \{z : Øz\} \equiv f\{z: \psi z\}]$ (*20.18) etc.

Russell then shows that classes of classes satisfy all these properties also.

Number in terms of Class [Mukhopadhyay, A. K. (2015)] ***:***

We know that arithmetic is all about numbers. Numbers are of two Kinds — *cardinal numbers* and *ordinal numbers*. Informally speaking, a cardinal number is the number that we speak of in answer to the question "how many"? It is the number indicating the strength of a set/class. On the other hand, ordinal numbers are numbers that we speak of while counting the elements/members of a set as the first, the second, the third, and so on.

In mathematics, there are mainly two traditions of defining numbers. One is the Frege-Russell tradition of defining numbers as classes of similar classes/sets of equivalent sets; the other tradition goes back to Dedekind and also to Peano, in which fundamental properties of numbers are given in the form of some axioms/primitive propositions.

Russell has defined cardinal numbers as equivalence classes of classes; and ordinal numbers as equivalence classes of well-ordered classes of the same type, in accordance with their respective logicist programs.

The number of a class is the property that belongs to the class collectively and not distributively. Definition of number by abstraction as some common property shared by similar classes does not satisfy the condition of uniqueness. This definition does not guarantee that there is exactly one common property shared by two or more similar classes.

To avoid this problem Russell defines numbers as classes of similar classes. They are unique in respect of their extensions. A cardinal number, i.e., the cardinality of a given class is the class of all classes similar to the given class.

Thus, Russell's definition of the number zero is the class whose only member is the null class.

The definition of the number one is the class of all singletons, and the definition of the number two is the class of all couples, and so on.

In general, a number is any thing which is the number of some classes.

Since, the number of a class has already been defined without reference to 'number', the question of circularity does not arise.

Unlike definitions of numbers by abstraction, Russell's definitions of numbers as classes of classes ensure that each particular number is unique. Because, each particular number, according to Russell, is identified with a class (of similar classes) that is identical only with itself. If there is another class of similar classes to be identified with a particular number, then by the principle of extensionality this second class would be identical to the first class.

The above definition of a cardinal number given by Russell is the definition of a particular finite number. It remains to be seen how the series or progression of natural numbers, i.e., 0, 1, 2, 3,….., and also infinite cardinal numbers are to be defined by Russell.

Peano's Postulates for Natural Numbers [Mukhopadhyay, A. K. (2015)]. ***:***

Peano encapsulates the whole of the theory of natural numbers with the help of three primitive ideas — 'zero', 'number' and 'successor', and five postulates.

Let '0' mean X_0, 'number' mean the whole set W of terms, and let 'successor' of any term X_n mean X_{n+1}. Then, we may express Peano's five postulates as follows —

A1) 0 is a number.
[That is, 0. is a member of the set W, i.e., $0 \in W$]

A2) The successor of any number is a number.
[That is, taken any term X_n in the set W, X_{n+1} is also in the set, i.e., for each $X_n \in W$, there exists a unique X_{n+1} ε W]

A3) No two numbers have the same successor.
[That is, if X_m and X_n are two different members of the set W, X_{m+1} and X_{n+1} are different, i.e., if X_m, X_n W such that $X_n \neq X_m$, then $X_{n+1} \neq X_{m+1}$.]

A4) 0 is not the successor of any number.
[That is, no term in the set W comes before X_0, i.e., there is no $X_n \in W$ for which $X_{n+1} = 0$.]

A5) Any property which belongs to 0, and also to the successor of every number which has the property, belongs to all numbers.

[That is, any property which belongs to X_0, and belongs to X_{n+1} provided it belongs to X_n, also belongs to all X_i's, $i \in I=W$,i.e., if S is a subset of W such that $X_0 \in S$, $X_{n+1} \in S$ for every $X_n \in S$, then $X_i \in S$ for all $i \in I = W$.].

The above five postulates give the fundamental property of a progression or, a series of the form — X_0, X_1, X_2 ,…, Xn , … .

In any series of the above form, there is a first term, a successor to each term (so that there is no last term), no repetitions, and every term can be reached from the start in a finite number of steps. Every progression, according to Russell, is a series that verifies these five postulates. 'Zero' is given the name to its first term, the name 'number' to the whole set of its terms, and the name 'successor' to the next in progression.

The fifth postulates in particular, which is known as the *Principle of induction* gives the nature of a progression.

Russell's Definition of Cardinal Numbers Satisfying Peano's Postulates [Mukhopadhyay, A. K. (2015)] ***:***

Now, it is to be seen how Russell captures the notion of progression of natural numbers in his theory.

Peano's three primitive notions are given in Russell's theory through definitions. Cardinal number is defined as the number of a given class. Each particular number is an instance of cardinal number. Zero is defined as the cardinal number of the class consisting only of the null class of a given logical type.

$$0 = \{\Lambda\} \text{ Df.}$$

Let us now define the notion of 'successor', following Russell.

"The successor of the number of terms in the class α is the number of terms in the class consisting of αtogether with x, where x is any term not belonging to the class."[7] [Russell, B. (1919)]

It can be shown that in Russell's theory we get n+1, i.e., the class of all classes having n+1 terms as the successor of n, i.e., the class of all classes having n terms.

Let us take the number $0 = \{\Lambda\}$.

Then, by the above definition, the successor of 0, i.e., 1 is the cardinal number of the class $\Lambda \cup \{x\}$, where x does not belong to Λ.

In other words, $1 = \{\{x\}\}$. Whatever x may be, $\{\{x\}\}$ is the class of a class having only one member, and of any class equivalent to it.

The successor of 1 is 2. 1 is the cardinal number of the class {x}, whatever x may be. Then, 2 is the cardinal number of the class $\{x\} \cup \{x'\}$, where x' does not belong to {x}. Thus, $2 = \{\{x, x'\}\}$. That is, 2 is the class of a couple.

In this way, it can be shown that 3, which is the successor of 2, is the class of a trio, and so on. In general, the notion of successor would give n+1, i.e., the class of all classes having n+1 terms, as the successor of n, i.e., the class of all classes having n terms.

Thus, Peano's first two postulates come through in Russell's theory of classes, in which numbers are defined in terms of classes.

Now, we would try to understand how Peano's fourth postulate is also available in Russell's theory of classes.

Let us recall that 0 is the class {Λ}. Also, suppose that 0 is the successor of some number k. Then, 0 is the cardinal number of the class composed of k number of elements together with any x that is not a member of the class of k elements. Then, 0 becomes the cardinal number of the class consisting at least of x. This implies $x \in \Lambda$which, we know, is false. So, it is not true that there is some number k such that 0 is the successor of k.

The fifth postulate of Peano is given by a definition in Russell's theory. But, before stating that definition we have to understand the notion of posterity first. The posterity of a given natural number with respect to the relation "immediate predecessor" (which is the converse of "successor") is all those terms that belong to every hereditary class to which the given number belongs. A hereditary class, in its turn, is a class having the successor of n, that is, n+1 as its member whenever n is a member of that class, for any n.

Now, let us present, following Russell, the fifth postulate of Peano, that is, the *principle of mathematical induction*. The postulate is —

The "natural numbers" are the posterity of zero with respect to the relation "immediate predecessor" which is the converse of "successor".

It is not difficult now to understand how any assigned natural number can be generated from zero by successive steps from “next to next”.

Thus, in Russell’s theory, cardinal number is defined first, and then a natural number is defined as a cardinal number satisfying the principle of induction. In fact, a natural number is a finite cardinal number. By the principle of induction, all natural numbers given by Peano’s axioms are generated in Russell’s theory of classes. The collection of these natural numbers is an inductive class of which 0 is a member and if any natural number n is a member of this class, then its successor n+1 is also a member of this class. In other words, the numbers forming such an inductive class are *inductive numbers*.

Now, if the process of generating natural numbers by successor function is to generate numbers infinitely, then the third of Peano’s postulates, namely, ‘no two different numbers have the same successor’ must hold good.

However, this can be ensured only if the totality of objects in the universe is assumed to be infinite. The *axiom of infinity* is just this postulate. According to this postulate —

“*if n be any inductive cardinal number, there is at least one class of individuals having n terms*.”[8] [Russell, B.(1919)].

Assuming Peano’s third postulate to hold good, it can now be said that the class of inductive numbers is an infinite class. Then the cardinal number of this class can not be one of the inductive numbers, it must be something new. For example, the number of elements in a series starting from 0 to n is obviously none of 0, …, n, but n+1. Thus, the cardinal number of the class of inductive numbers is a new number, say ω(omega), which is none of the finite inductive numbers 0, 1, 2,… .

ω(omega) is the first *transfinite cardinal number*, which is the class of all classes similar to the class of inductive or natural numbers. Other transfinite cardinal numbers are defined accordingly in Russell’s theory.

Gödel’s Incompleteness thesis and Russell’s Logicist Program:

It is generally thought that Gödel’s proofs become the death sentence for Russell’s program; however, we can have a different view, as we can find some Flaws in Gödel's Incompleteness Thesis.

Most significant part of Gödel's paper[12] is his Fifth Proposition.

The entire result of Gödel's thesis depends on this Fifth Proposition. Since it is the crux of the proof, it is expected that he would go into it in great detail, and would offer a proof of this proposition in a very clear and logical fashion. However, he only provides a brief outline of how one might construct a proof.[16] Now, there is nothing wrong in having a new idea, but to rely on a brief outline instead of giving a rigorous and complete proof is simply not good enough for a revolutionary claim. To assume that a theorem is correct because you have a gut feeling that it is correct simply is not acceptable.

Let us consider the following:

1) **'Every thing is a *C*',**
2) **‘Arnab is a *C*'.**

All that we have done here is to substitute the variable 'thing' by the name ***'Arnab'***, which designates a value of that variable and, removed the quantifier ***'Every'*** to arrive at new expression, ***'Arnab is a C’.*** When we use ***'Arnab'*** in place of the variable ‘thing’ in **'Every thing is a *C*'**, ***'Arnab'*** is obviously not a variable, but designates a specific person (me).

This simply shows that whenever there is a word / symbol which is a variable of a language, any value that it represents is not a variable, but a specific value in that language. When we have meta-languages and sub-languages (i.e. object languages) in our discussion, a variable in one language can be a specific value in another language. However, in that case, we have to be very careful that we do not *mix up* our languages.

In Gödel's proof, number relationships are matched with formal sentences, the statement of which is Gödel's Fifth Proposition.

The proposition states:

'For any number relationship with one free variable, there is a matching formal sentence with one free variable, where that formal sentence expresses the same concept as that number relationship.'

Since, ***'number relationship'*** is quantified by ***'for any,'*** this means that ***'number relationship'*** is a variable in the language of that proposition. That language is Gödel's proof language, i.e. a meta-language. This further implies that the specific values to which the variable ***'number relationship'*** refers to are specific number relationships. In other words, in the language of Gödel's proof, any specific number relationship is a specific value.

Again, the word ***'variable'*** in that proposition is itself a variable in the language of that proposition. The specific values that it refers to are symbols that are variables of number relationships. So, any symbol that is a variable of a number relationship is a specific value in the language of Gödel's proof.

In view of the above, we can say that in the language of Gödel's proof, number relationships are specific values, and variables of number relationships are also specific values. This means, the language of Gödel's proof has to be a language that is a meta-language to number relationships, and whatever language a number relationship might be expressed in, is a sub-language with respect to that meta-language.

Now, remember, Gödel's proof language is a meta-language with respect to the formal language in which the elementary number theory including the number relationships can be formalized. So, in that meta-language, formal sentences are specific values, and the variables in those formal sentences are also specific values.

Therefore, the language of Gödel's proof is a meta-language to *both* the formal language and to the language of number relationships. Gödel's meta-language is a meta-language *that talks about* formal sentences and *talks about* number relationships.

The problem is, we may argue that in his proof, Gödel confuses between his meta-language and the language of number relationships. Although the language of number relationships is a sub-language, Gödel makes the mistake of assuming that number relationships can also actually be valid expressions of his meta-language, but they cannot, because of the statement of his Fifth Proposition.

Since the formal language here is a sub-language, this means that formal sentences are not expressions of the meta-language. They are simply combinations of symbols that the meta-language *talks about*. Exactly the same argument applies to number relationships. Statement of number relationships cannot be expressions in that meta-language either – they are simply combinations of symbols that the meta-language *talks about*. So, we may argue that, neither statements of number relationships, nor formal sentences expressing them are expressions in the meta-language (i.e. Gödel's proof language), these are simply combinations of symbols with respect to meta-language. Hence, a statement about number relationship has no meaning in the meta-language.

In order that Gödel's proof language can translate a number relationship into a formal sentence, it has to use such variables as could translate expressions in ordinary language into symbols that make up the formal language, and which in its turn would translate number relationships. Because Gödel's proof language uses its own variables to refer to those values (number relationships) , any value that is one of those specific values cannot be a variable of the meta-language. So, the variables of formal sentences cannot be variables of the meta-language; nor can be variables of number relationships.

Once you realise that Gödel's proof language is a meta-language to the language of number relationships as well as to the formal system, we can show that Gödel's proof cannot possibly give the result that Gödel intended (if it is to be a logical proof) of it.

To effectuate the proof of his first theorem, Gödel defines the Basic numbering function so that it would always give the same value as the Gödel numbering function. He maintains:

'For any number, the Basic Number of that number is the same as the Gödel Number of that number'.

We can also state that as:

*'For every **x**, where **x** is a number, **BN**(**x**) = **GN**(**x**)'.*

Now, the question is, as '***BN*** (...)' and '***GN*** (...)' are defined as relationships, what language do they belong to?

Let us look at the Gödel Numbering function, '***GN***(***x***)'. The values that the free variable '***x***' can take are symbols of the formal language, or combinations of symbols of the formal language. That is, '***GN***(***x***)' is neither an expression in the language of number relationships, nor an expression in the formal language; it is an expression of the meta-language. That is, the variable '***x***' in '***GN***(***x***)' is a variable of the meta-language.

Now, let us look at the Basic Numbering function, **'*BN*(*x*)'**. This has to be a number relationship, and the variable 'x' in **'*BN*(*x*)'** is a variable of a number relationship.

Thus the proposition, *'for every* ***x***, ***BN***(***x***) = ***GN***(***x***)*'* is nonsensical because it **mixes up** the meta-language and its sub-languages. In this proposition, expression, the variable '***x***' is at the same time a variable of the meta-language and a variable of a sub-language – the language of number relationships. This means, the proposition above is not a legitimate wff in Gödel's proof language.

That shows that Gödel's proof, which uses the basic numbering functions to get his "true but unprovable" sentence will not go without contest.

Thus, Gödel's proof is under suspicion and certainly cannot be taken as a death sentence to the system of *Principia Mathematica* and related systems.

Further, in any philosophical argument it is required that the opponent should argue against the proponent of a thesis by keeping proponents thesis intact.

But, the way Gödel presented his theory of class (or, set) is not the same as that Russell proposed.

For example, One of the intentions of Russell to introduce the theory of logical types was to prevent self-referential sentences; however, by introducing Gödel Numbering, Gödel opened up a clear option to admit self-referential sentences in his proposed formal system that represents any formal system capable of expressing elementary number theory and with respect to which Gödel proceeded to prove his incompleteness thesis. This is unfortunate.

Again, by means of Gödel Numbering, a sentence like 'Arnab is a fool' (like, "2+2=4") and a sentence of different level like 'It is true that Arnab is a fool' (like, "It is true that '2+2=4'") would get Gödel Numbers of the same nature (though, the numbers may be different), which is not tenable from the philosophical point of view.

Also, it is certainly not intelligible to make a claim of a system, which is more complex than the inherent complexity of that system. For example, a television set has some complexity by which it functions, but it is not possible for that television set to perform anything beyond its capacity. Similarly, a formal system (incidentally able to contain elementary number theory) is constructed with syntax and semantics of specific potency to prove theorems about its specific object of study. But to prove its own soundness, completeness etc. is proving more complex claims than the complexity of the system itself. These claims are about the system, taking that system a sub level object of study, no matter if there is a mechanism to do so.

Moreover, as a philosopher of mathematics, it is difficult for Gödel to say anything about the nature of number convincingly and the ground for the necessity of mathematical truth formally after proposing his incompleteness thesis, since the same incompleteness argument can be triggered against Gödel's own system.

Conclusion

Now, given that Gödel's threat becomes lessen, we can say that-

- Mathematics deals with numbers and since numbers are incomplete symbols, nature of number as objective entity does not arise.
- What matters is the meaningfulness (i.e. truth, falsity) of number statements.
- Since mathematical notions are shown to be logical notions by definition and mathematical truths are shown to be logical consequences of a few logical truths, they are necessarily true.

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